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# AN ALGORITHMIC APPROACH TO IMPLEMENT GRACEFUL LABELED CYCLE-STAR GRAPH

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**Abstract.** Graph theoretic solutions aids to a greater extent in solving many real-life practical problems. Graph labeling has become a popular research interest for a broad range of applications in recent era due to its disambiguated trait. Graceful labeled graph selects distinct vertex labels which in turn induces distinct edge labels. Unlike directed graph, we choose to study on simple undirected graph as it harmonizes well with social network behavior. In this paper, we present an algorithm to implement a unicycle graceful graph from star copies on cycles. Also we illustrate the algorithmic implementation using igraph graph analysis in R toolkit. We infer representing a graph model irrespective of any number of vertices and edges maintains the graceful graph novelty.
**Keywords.** Graceful labeling, Cycle graph, Star graph, igraph R network, tkplot visualization, Fruchterman-Reingold layout.

## 1. INTRODUCTION

Graph labeling is an active research area in graph theory with multidisciplinary research applications. The concept of graceful label was first introduced by A.Rosa [1] by the name of β –labeling which leveraged many practical applications in the field of radar communication, facility graph, wireless local area network, circuit design, worm propagation, Social Networks, Compression Networks, packet routing etc . However, the fundamental understanding that the characterization of graceful and other labeled graphs appears to be one of the most difficult and hard problems in graph theory [5]. Rosa identified that graph having many vertices without many edges and too many edges and certain dissimilarity fails to be graceful. Therefore, many labeling schemes have been introduced and they were explored by many researchers. Some persisting labeling schemes are Harmonious, prime, cordial, Magic and Antimagic which is applicable to undirected finite bipartite graphs. A consolidated summary of graceful and non-graceful results along with some unproven conjectures can be found in Gallian’s dynamic survey.[2]. Most study in graceful labeling chooses to solve a simple graph i.e. graph without loops or parallel edges. [3].

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Seoud and Youssef [8] have proved the join of any two stars is graceful and the join of any path and any star is graceful. K.M. Koh et al[9] further proved the join of cycle and broken path $C_3 + P(n, t)$ is graceful for all $n \geq t + 2, m \geq 3$ $and$ $1 \leq t \leq 3$. T.-M. Wang, G.-H. Zhang [7] has introduced a notion of edge graceful deficiency on simple finite graphs. The Parameters that violate the edge-gracefulness of cycles is studied. Also parameters involved in graceful properties of simple undirected graph are studied [11]. Kaneria et al. [6] proved that resultant graph obtained by interconnecting the star graph is found to be graceful. Also interconnecting the cycles is found to be graceful [3]. Kaneria et al. [14, 17] proved that interconnecting cycles $C_m; C_n$ by an arbitrary path is graceful on $m, n \equiv 0 \ (mod\ 4)$ as well as cycle of cycles is graceful
Santhakumaran, A. P., & Balaganesan, P. [4] proved that a graph $lA(m_j, n)$ is vertex-graceful for both $n$ $and$ $l$ odd, $0 \leq i \leq n - 1, 1 \leq j \leq m_i$ . Rosa proved that an Eulerian graph with number of edges $m \equiv 1 \ (mod\ 4)$ $or$ $m \equiv 2 \ (mod\ 4)$ cannot be graceful [1]. Also, Rosa proved that the cycle $C_n$ is graceful if and only if $n \equiv 0 \ (mod\ 4)$ $or$ $n \equiv 3 \ (mod\ 4)$.
Sethuraman [10] present an Embedding Algorithm that generates a graceful unicyclic graph from any acyclic graph whose results agrees to Truszczynski's Conjecture. Backtracking and probabilistic search algorithm [12] is verified for graceful tree conjecture by testing at most 35 vertices. It does not guarantee to produce a graceful labeling and loops infinitely. Reductionist algorithm was proposed to improvise the computational graceful labeling approach in trees [13]. A sequential algorithm [15] is demonstrated in linear runtime to perform odd graceful labeling of vertices and edges in recursive manner. A fuzzy inspired graceful labeling [16] is proved to label 1241 edges on a triangular star graph.
A graph labeling scheme is practically suitable to simplify a graph to a defined form. The criterion for any basic labeling algorithm is to focus on appropriate labeling pattern that best reduces the label size contributing in cost efficiency. The motivation of this work is to facilitate the generation of graceful cycle-star graphs to depict the real world network connectivity.
The paper is organized as follows: We first present the preliminaries in Sect. 2. Next in Sect. 3, we present a novel algorithm to perform Simple Cycle graceful labeling and Cycle-Star graceful labeling algorithm. In Sect. 4, we exclusively illustrate vertex labels using R graph analysis and visualize the graceful graphs. We finally conclude the paper with some promising future directions in Sect. 5.

## 2. PRELIMINARIES

Given a graph G with $E_G$ edges, a labeling of the nodes with distinct integers from the set {0, 1, 2... $E_G$} induces an edge labeling where the label of an edge is the absolute difference of the labels of the nodes incident to that edge.

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**DEFINITION [17] 2.1.** A function $\mathrm{f}$ is called *graceful labeling* of a graph G if $\mathrm{f} : \mathrm{V} \rightarrow \{0, 1, 2, \ldots, \mathrm{q}\}$ is injective and the induced function $\mathrm{f}^* : \mathrm{E} \rightarrow \{1, 2, \ldots, \mathrm{q}\}$ defined as $\mathrm{f}^*(\mathrm{e}) = |\mathrm{f}(\mathrm{u}) - \mathrm{f}(\mathrm{v})|$ is bijective, where $\mathrm{e} = (\mathrm{u}, \mathrm{v})$. A graph which admits graceful labeling is called *graceful graph*.

**DEFINITION [18] 2.2.** The star graph $S_n$ of order $'n'$ is a tree on $'n'$ nodes with one node having vertex degree $(n-1)$ and the other $(n-1)$ node having vertex degree 1. Star is a complete bigraph $K_{1,n}$.

**DEFINITION [3] 2.3.** A cycle $C$ in a graph is a connected subgraph in which the degree of every vertex in $C$ is two. A cycle is a closed walk with n vertices denoted by $C_n$.

**DEFINITION [18] 2.4.** A path in a graph is an open walk in which no vertex is repeated. A path of length $(n+1)$ is called n-path and is denoted by $P_n$. Path length is measured as total count of edges interlinking any two different nodes.

**DEFINITION [3] 2.5.** The graph $C_n * nS_n$ is a $(n^2+n, n^2+n)$ connected graph such that every vertex of a circuit $C_n$ is identified (merged) with the root of a star $S_n$. It is a representation of equal distribution of star copies of cycle. The graph $\mathrm{C_n * nS_n}$ is graceful when n ≡ 0 or 3 (mod 4). Here n represents equal distribution of star and cycle nodes. The number $\{n2 – [(k-1)n + k]\}$ is not labeled for $n \equiv 0 \ or \ 3 \ (mod \ 4)$, and for some $k = 1,2,3, \ldots$

## 3. VARIATION OF GRACEFUL LABELLING

In this section, we present the algorithm to perform Cycle graceful labeling and Cycle-Star graceful labeling. The integration of star copies of cycle irrespective of any number of vertex selections under graceful property is introduced.

**DEFINITION 2.6.** An inequality based distribution of Star copies on cycle is represented as $C_{n_c} * n_c S_{n_s}$ graph. Here $n_c$ and $n_s$ represents count of cycle and star nodes. It takes $[(n_s * n_c) + n_c, (n_s * n_c) + n_c]$ connected graph such that every vertex of a circuit $C_{n_c}$ is merged to the root of a star $S_{n_s}$ . Every Star copies on cycle graph exhibits interconnecting network behavior which strongly favours community selection. When $n_c = 8$, the gracefulness of graph $C_8 * 8S_5$ with 48 vertices and 48 edges, is labeled. A Simplified label assignment pattern for the same is shown in equation 1.

$$
\begin{aligned}
&V_{6n} < f(V_0) < V_{6n-4,} \\
&V_1 < f(V_{6n-5}) < V_{5,} \\
&V_{6n-6} < f(V_6) < V_{6n-10,} \\
&V_7 < f(V_{6n-11}) < V_{11,} \\
&V_{6n-12} < f(V_{12}) < V_{6n-16} \ \ldots \\
&V_{(6n/2)-5} < f(V_{(6n/2)}) < V_{(6n/2)-1}
\end{aligned} \qquad (1)
$$

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All the vertices are labeled in clockwise direction starting from $V_0$. For $n \equiv 0 \ (mod\ 4)$, $9\text{n}/2$ is not used in the graceful labeling. For $n \equiv 3 \ (mod\ 4)$, $9(n+1)/2$ is not used in the graceful labeling. For $n = 8$ and $k = 2$, the number $36$ is not labeled. The illustration for $C_8 * \ 8S_5$ graph is shown in Figure 1.

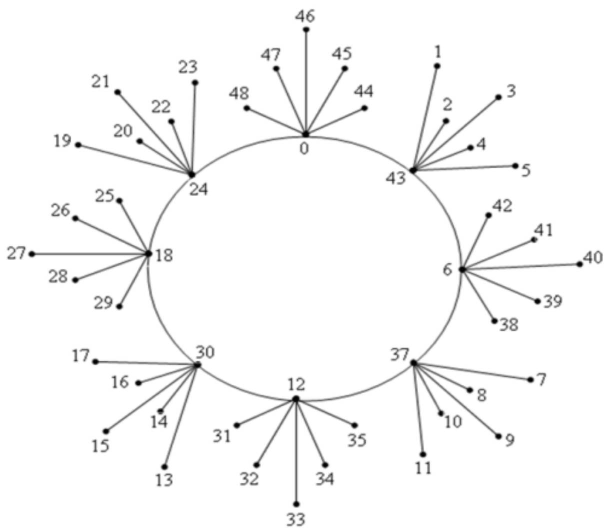


Figure 1. $C_8 * 8S_5$ graph .Distinct node distribution of star copies of cycle for $n \equiv 0 \ (mod\ 4)$

### 3.1 GENERALIZED ALGORITHMIC FRAMEWORK

We focus to generalize the algorithm in two stages; first to construct a cycle graph satisfying the gracefulness for n number of nodes and second to construct star replicas merged to cycle graph. In order to facilitate the generation of graceful labeled graph, the labels are transformed in adjacency list format. Algorithm1 is traced in sequential manner to serve this purpose. The labeling of vertices in the cycle $C_n$ for $n$ desired vertices is listed. By default, the star node $S_n$ is nullified. The algorithm runs in single pass in which cycle nodes $C_n$ is represented as $v_1, v_2, v_3, \ldots, v_n, v_1$ . We consider showing that $C_n$ is graceful if and only if $\equiv 0 \ (mod\ 4)$ . Figure 2 shows the illustration of graceful labeled cycle graph for n=8 .We assign label $v_i$ when [1],

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$$v_i = \begin{cases} \frac{i-1}{2}, if\ i\ is\ odd \\ n+1-\left(\frac{i}{2}\right), if\ i\ is\ even\ and\ i \le (\frac{n}{2}) \\ n-\left(\frac{i}{2}\right), if\ i\ is\ even\ and\ i > \left(\frac{n}{2}\right) \end{cases} \qquad (2)$$

**Algorithm 1: Algorithm for Cycle graceful labeling**
Step 1: Get input for cycle nodes
Step 2: Find midcycle:= cycle/2
Step 3: Initialize local variables for last cycle and last index
lcycle =cycle; li = 0
Step 4: for loop to cycle i as integer = 1 to cycle
Step 5: check i < midcycle = False ; lcycle = lcycle -1 ; set flag midcyclestart = true;
Step 6: check if i = 1; print (0, cycle)
Step 7: else if i Mod 2 = 0; check If cycle = i; print (lcycle, 0) else li = li – 1 , print (lcycle, li);
Step 8:else lcycle = lcycle – 1, print (li ,lcycle);
Step 9: Stop

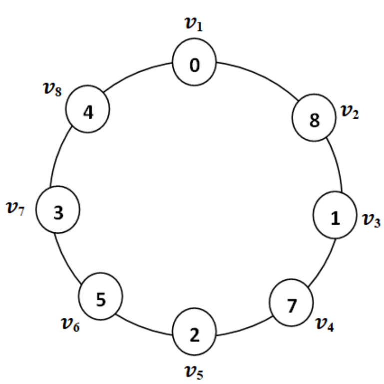


Figure 2.Graceful labeling of Cycle graph $C_8$

For $\mathrm{n} \equiv 0 \ (\mathrm{mod}\ 4)$, $C_n = (3\mathrm{n}/4)$ is not used in the graceful labeling. For $n = 8$, $C_8 = \left(\frac{3*8}{4}\right) = 6$ is unlabeled as shown in figure 2. With some modification we can use the above algorithm to represent cycle graph for odd values of n such that $n \equiv 3\ (mod\ 4)$. Algorithm 2 generates cycle-star graph such that $\mathrm{n} \equiv 0\ (\mathrm{mod}\ 4)$. The adjacency list for the labeling pattern is sequentially created by mapping $(cycle, star)$ pairs and $(cycle_i, cycle_j)$ pairs. With reference to definition 2.6, the algorithm uses variables count, outvalue to represent the set of connected pairs and unlabelled node value

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respectively in $C_{n_c} * n_c S_{n_s}$ graph. For $\mathrm{n} \equiv 0 \text{ or } 3 \pmod 4$ and $(cycle - star) = 3$, then graceful labeled graph can be constructed. The runtime efficiency of algorithm 1 is $O(\log n_c)$ while algorithm 2 is $O(n_c * n_s)$.

**Algorithm 2: Algorithm for Cycle-Star graceful labeling**
Step 1: Get input for cycle nodes and star nodes.
Step 2: Initialize local variables count, outvalue, lodd, leven, str as string
Step 3: Assign values for the local variables count = (cycle * star) +cycle;
lodd=0;leven = count; str = emptystring;
Step 4: if (cycle-star) =3 then;
if cycle Mod 4 =0 then; outvalue = (9*cycle) /2;
elseif cycle Mod 4 =3 then outvalue = [9*(cycle+1)] /2; end if
Step 5: for loop to cycle i as integer = 1 to cycle
Step 6: check if i =1; for loop to cycle j as integer = 1 to star;
if j>1 then; leven =leven -1 if outvalue = ieven then; leven =leven -1endif
print(lodd & " || " & leven)
str1 = lodd
Step 7: check else if i Mod 2 = 0 ; for loop to cycle j as integer = 1 to star;
If j > 1 Then; lodd = lodd+1 if outvalue = lodd then; lodd = lodd +1 endif
print(leven & " || " & lodd)
str1 = str1 & "," & leven
Step 8: else for loop to cycle j as integer = 1 to star;
if j>1 then; leven =leven -1 if outvalue = leven then; leven =leven -1 endif
print(lodd & " || " & leven)
str1 = str1 & "," & lodd
Step 9: end loop
Step 10: declare a string array as str assign str1.split
Step 11: for loop to cycle k as integer = 1 to string. Length
Step 12: local variable declaration val1 and val2
Step 13: assign local variable val1 = str(k) and val2 =0
Step 14: check if k < str.Length – 1 then val2 = str(k + 1)
Step 15: print val1 and val2
Step 16: Stop

## 4. IMPLEMENTATION AND RESULTS

The proposed algorithm is executed using R language. In this process, a comprehensive R archive network package igraph is used for graph analysis and visualization. In this section, the graceful labeled representation of both Cycle graph and Cycle-Star graph is visualized.

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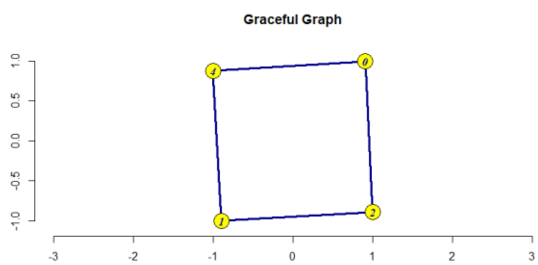
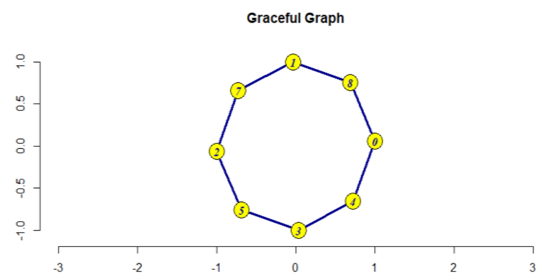
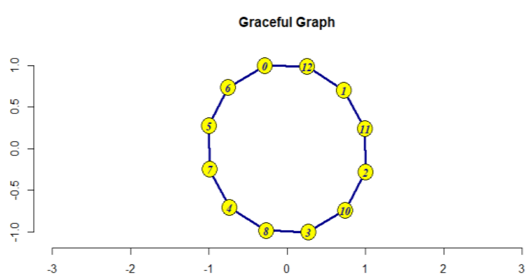
## 4.1 Cycle graceful graph generation



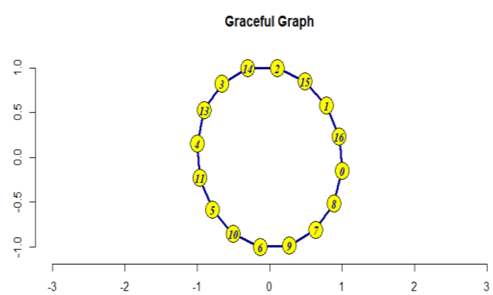

***(a)***




***(b)***




***(c)***




***(d)***

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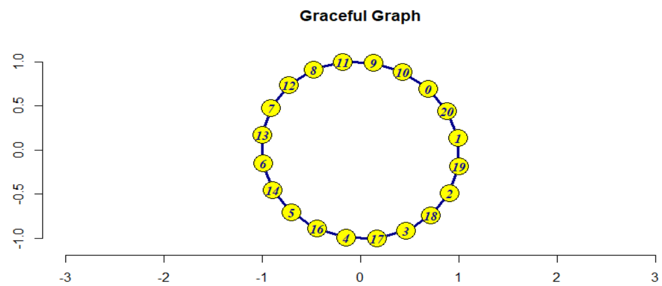


***(e)***

Figure 3. Visualization of Cycle graceful graph. (a) $C_4$ (b) $C_8$ (c) $C_{12}$ (d) $C_{16}$ (e) $C_{20}$

The illustration of graphs is shown in figure 3. Some samples are presented according to algorithm1 satisfying only if $\mathrm{n} \equiv 0 \pmod 4$. The cycle graph $C_4$ has vertex label set {0, 1, 2, 4} and label 3 is not labeled. The edge labels identified for this graph is {1, 2, 3, 4}. Similarly for graph $C_8$ , $C_{12}$, $C_{16}$, $C_{20}$ the unlabeled values are 6, 9, 12, 15. Therefore it proves that for every $C_n$ , the label $(3n/4)$ is eliminated to maintain the gracefulness. R is most definite toolkit that serves to provide better graph network analysis and visualization. The popularity of R usage in graph is supported through well defined library packages. In the implementation process, igraph and qgraph library package was a convenient tool of choice. We make use of R data frame representation to implement the adjacency list of graph. To visualize the cycle graph, the circle plot is most primary illustration choice.

## 4.2 Cycle-Star graceful graph generation

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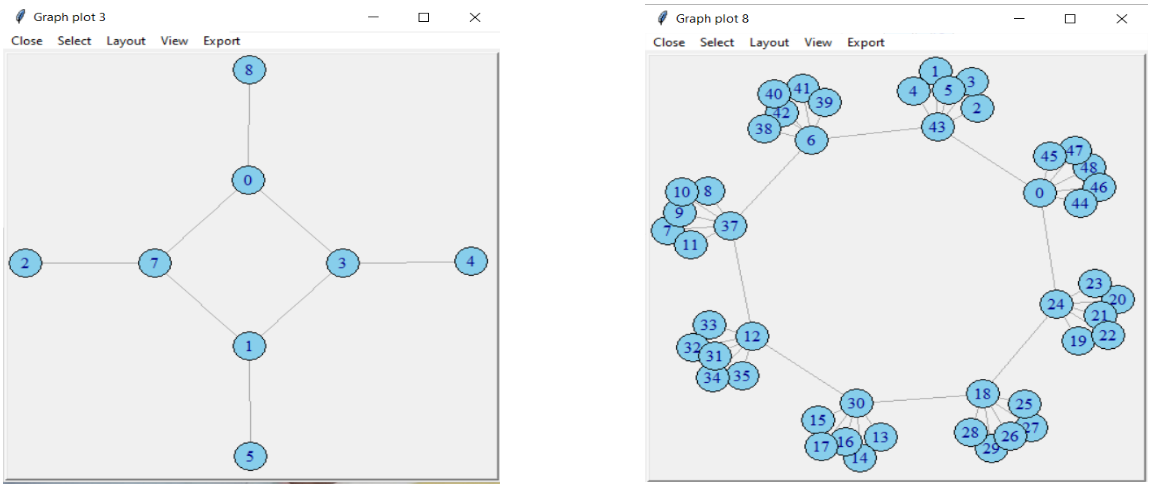


***(a)*** ***(b)***

Figure 4. Visualization of Cycle-Star graceful graph using Fruchterman-Reingold layout. (a) $C_4 * 4S_1$ (b) $C_8 * 8S_5$

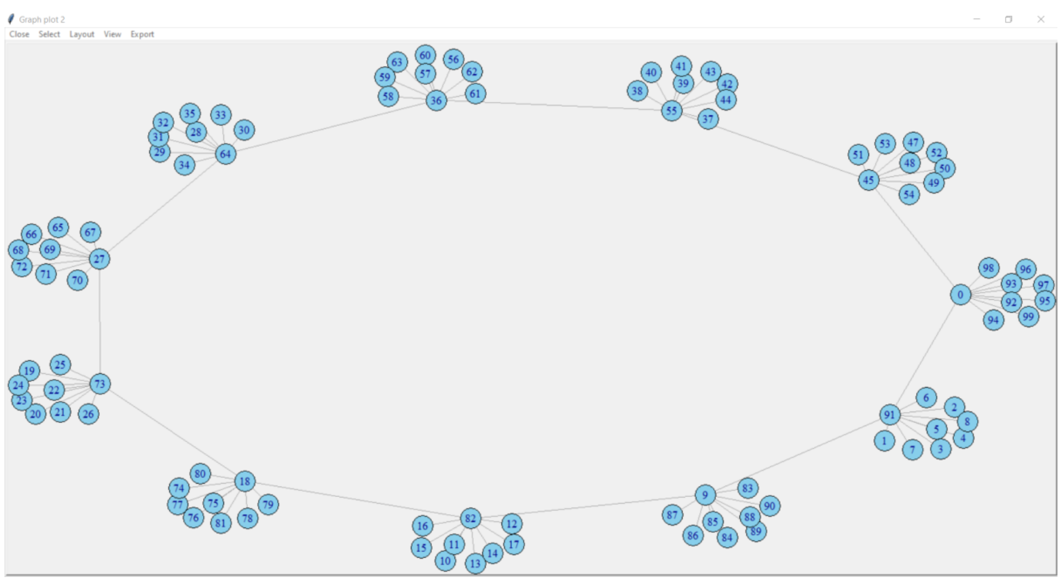


Figure 5. Visualization of Cycle-Star graceful graph $C_{11} * 11S_8$ on $n_c \equiv 3\ (mod\ 4)$

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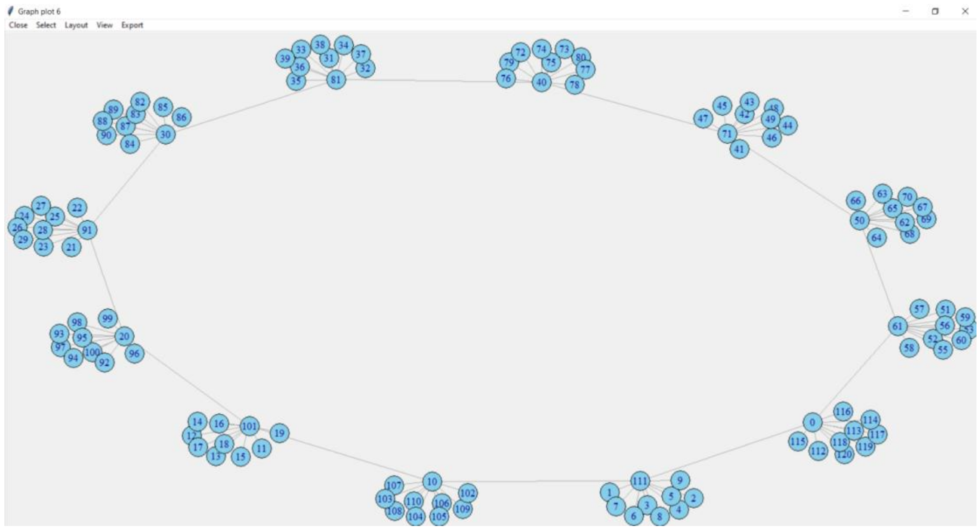


Figure 6. Visualization of Cycle-Star graceful graph $C_{12} * 12S_9$ on $n_c \equiv 0 \ (mod\ 4)$

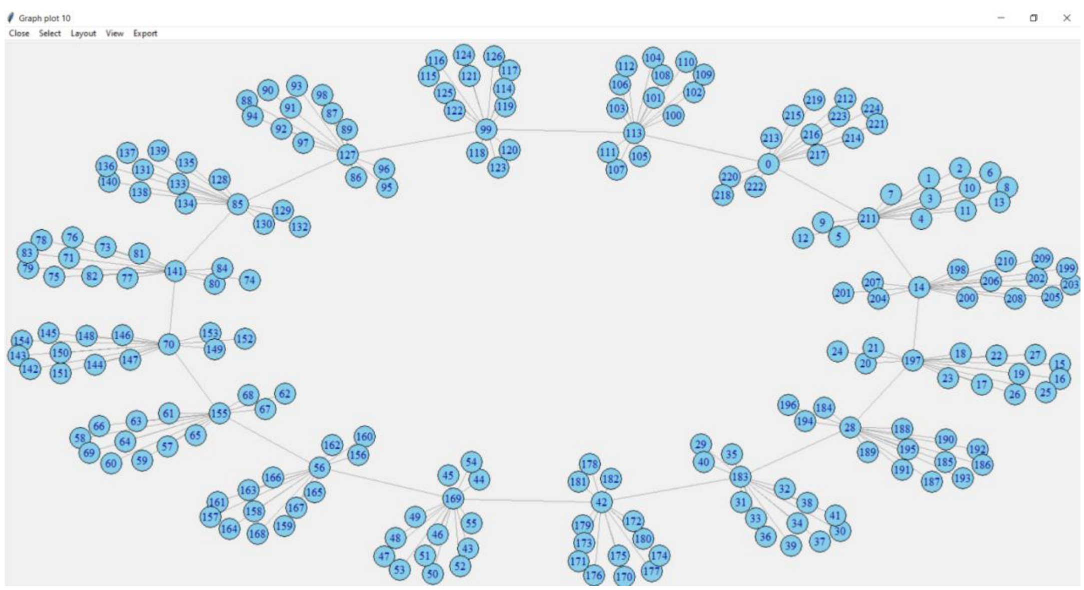


Figure 7. Visualization of Cycle-Star graceful graph $C_{16} * 16S_{13}$ using Kamada-Kawai layout

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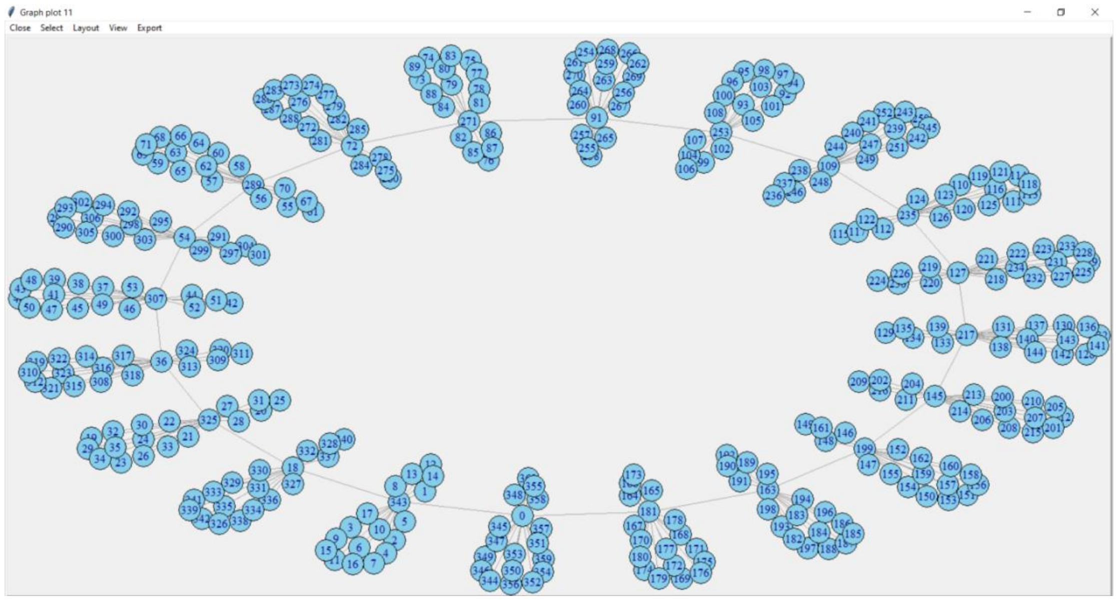


Figure 8. Visualization of Cycle-Star graceful graph $C_{20} * 20S_{17}$ using Kamada-Kawai layout

The Cycle-Star graph $C_4 * 4S_1$ has vertex label {0, 1, 2, 3, 5, 6, 7, 8} whose corresponding edge labels have to be {1, 2, 3, 4, 5, 6, 7, 8} in graceful case. As observed in figure 4a, edge labels are distinct corresponding to graceful labeling pattern. And also gracefulness is preserved in graph plotted from figure 4b to figure 8. This is to note that cycle- star graph $C_8 * 8S_5$ is selected as minimum threshold for labeling. Also graphs are visualized using Fruchterman-Reingold layout for significantly lower number of vertices. In order to visualize larger graphs, Kamada-Kawai layout is preferred for its precision. Figures 7 and 8 uses Kamada-Kawai layout for graph nodes $n_c \geq 16$ . As mentioned in algorithm 2, figures 5 and 6 show that visualization is possible when $n_c \equiv 0 \; or \; 3 \; (mod \; 4)$ .

## 5. CONCLUSION AND FUTURE WORK

We discussed a cycle-star graceful graph as an extension from classical cycle graph. We present an algorithmic framework to construct a graceful labeled cycle-star graph. R network analysis has extensive library functionalities, hence serves as a good visualization tool. The observations on implementation signify that cycle- star graph shows unique pattern maintaining desired graph novelty. The intension of this work is to promote visual programming of mathematical model representing real-world network behavior. Further we plan to experiment with suitable real-world dataset to mark its network performance. Also, in future we plan to implement the same for some special graphs.

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